CS342 - HW1 Solution

September 2018

1 Estimate Pi

Given $k \times 2$ samples $\{x_i\} \sim U\big[0, 1\big]$, estimate π.

1.1 Naive Estimator

The naive estimator given in the write-up uses k samples.

$$\begin{aligned} U(x, y) &= 1\{x^2 + y^2 < 1\} \\ \bar{U}(X, Y) &= \frac{1}{k} \sum_i U(x_i, y_i) \end{aligned}$$

Note this is indeed unbiased

$$\begin{aligned} \mathbb{E}(U) &= P(x^2 + y^2 < 1) \\ &= \frac{\pi}{4} \\ \mathbb{E}(\bar{U}) &= \mathbb{E}(\frac{1}{k} \sum_i U_i) \\ &= \frac{1}{k} \sum_i \mathbb{E}(U_i) \\ &= \mathbb{E}(U) \\ &= \frac{\pi}{4} \end{aligned}$$

1.2 Better Estimator

The better estimator uses $2k$ samples. It still does a monte-carlo estimate, but integrates one dimension in closed form.

$$\begin{aligned} U(x) &= \sqrt{1 - x^2} \\ \bar{U}(X) &= \frac{1}{k} \sum_i U(x_i) \end{aligned}$$

It is indeed unbiased

$$
\begin{aligned}
\mathbb{E}(U) &= \int_0^1 \sqrt{1-x^2}dx \\
&= \frac{\pi}{4}
\end{aligned}
$$

Another way to see why it is unbiased

$$
\begin{aligned}
\frac{\pi}{4} = \mathbb{E}_{xy}(U') &= \mathbb{E}_{xy}(1\{x^2+y^2<1\}) \\
&= \mathbb{E}_x\big(\mathbb{E}_y(1\{x^2+y^2<1\}|x)\big) \\
&= \mathbb{E}_x\big(\mathbb{E}_y(1\{y<\sqrt{1-x^2}\}|x)\big) \\
&= \mathbb{E}_x\big(\sqrt{1-x^2}\big) \\
&= \mathbb{E}(U)
\end{aligned}
$$

Why is it better? Let U be our better estimator, and U' the naive one.

1. U has a lower variance.

$$
\begin{aligned}
Var(U') &= \mathbb{E}(U'^2) - \mathbb{E}(U')^2 \\
&= \frac{\pi}{4} - \frac{\pi^2}{16}
\end{aligned}
$$

$$
\begin{aligned}
Var(U) &= \mathbb{E}(U^2) - \mathbb{E}(U)^2 \\
&= \int_0^1 1-x^2dx - \frac{\pi^2}{16} \\
&= \frac{2}{3} - \frac{\pi^2}{16}
\end{aligned}
$$

2. $\bar{U}$ uses more samples

$$
\begin{aligned}
Var(\bar{U}') &= \frac{1}{k}Var(U') \\
Var(\bar{U}) &= \frac{1}{2k}Var(U)
\end{aligned}
$$