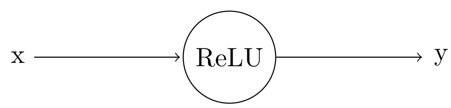
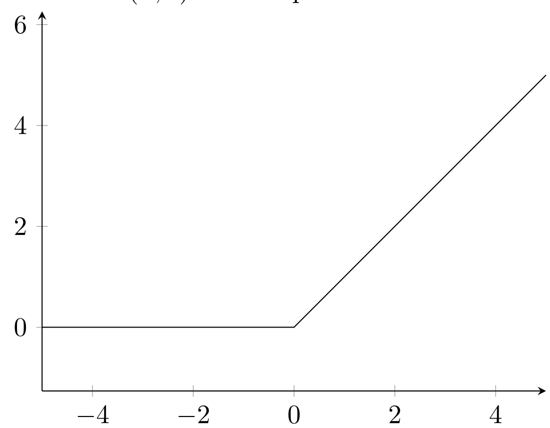
Exercise 1: Function approximation - Solution

Name: .... UTID:................

In this exercise you'll use linear functions and Rectified Linear Units (ReLUs) to build simple function approximators. You may work in teams of up to 3 students.

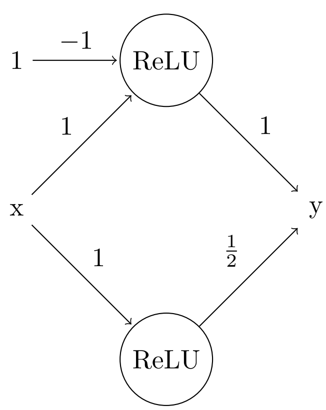
In all examples, we will approximate one dimensional functions: the input will be a single real number, the output another single real number. Formally, a ReLU is defined as $\max(x, 0)$ for an input x. It looks like this:

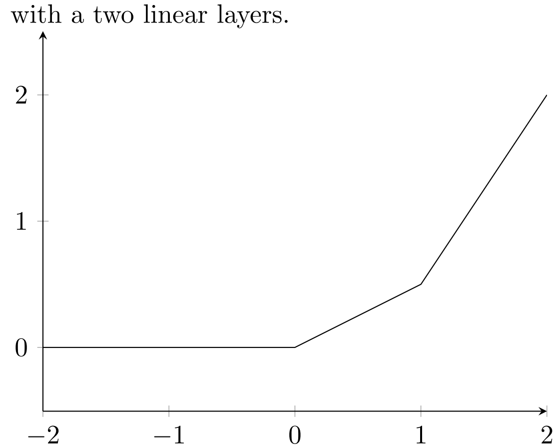




On the right you can see the corresponding network.

Now, let's see what happend if we combine a ReLU layer, with a two linear layers.





Here we subtract one from the input of the first ReLU, and divide the output of the second relu by two. The entire network corresponds to

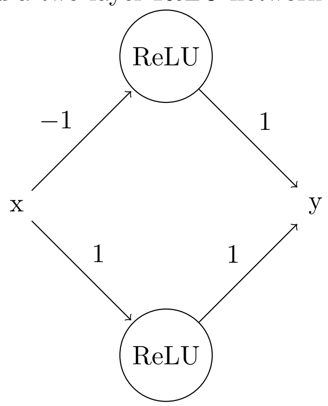
$$y = \max(x - 1, 0) + \frac{1}{2} \max(x, 0).$$

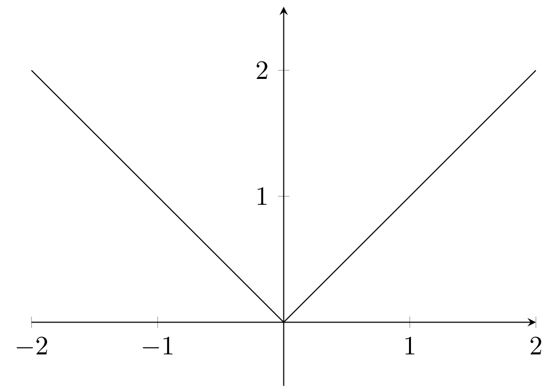
Note that we used a bias term in the linear layer here to add and subtract values.

Not let's start approximating functions.

1 abs

Build a two layer ReLU-network that approximates the absolute value $|x|$.

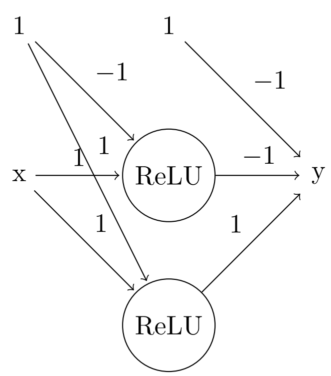


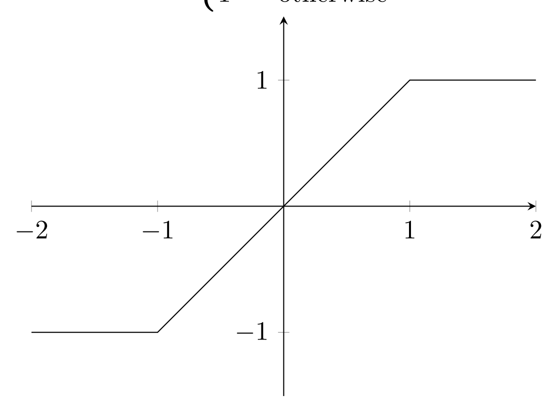


$$\max(x, 0) + \max(-x, 0)$$

2 soft step

Build a two layer ReLU-network that approximates the soft step function $y = \begin{cases} -1 & \text{if } x < -1 \\ x & \text{if } -1 \le x \le 1 \\ 1 & \text{otherwise} \end{cases}$.

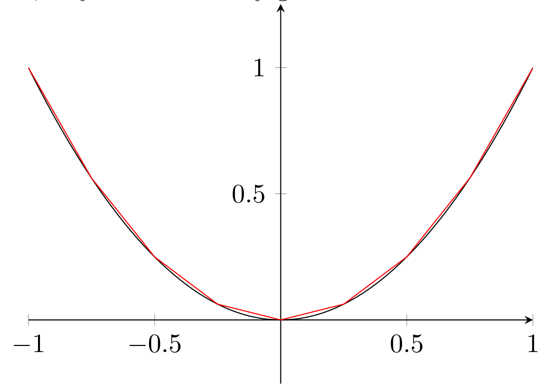




$$\max(x + 1, 0) - \max(x - 1, 0) - 1$$

3 Square

Build a two layer ReLU-network that approximates the soft step function $y = x^2$ in the range $[-1, 1]$ using 8 hidden units. This is the first function you'll have to approximate, as you will unlikely get an exact solution.



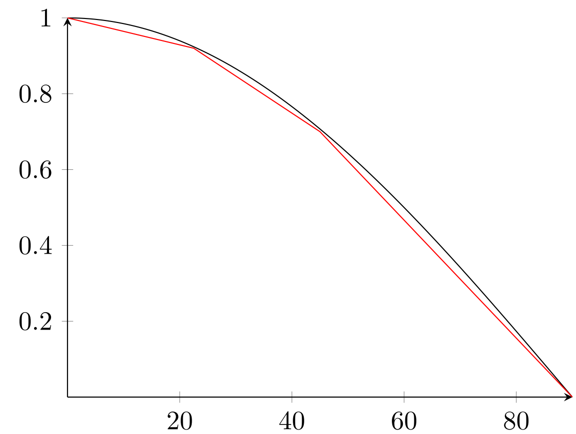
$$
\begin{aligned}
&\frac{1}{4}\max(-x,0) + \frac{1}{2}\max(-x-\frac{1}{4},0) \\
&+\frac{1}{2}\max(-x-\frac{1}{2},0) + \frac{1}{2}\max(-x-\frac{3}{4},0) \\
&\frac{1}{4}\max(x,0) + \frac{1}{2}\max(x-\frac{1}{4},0) \\
&+\frac{1}{2}\max(x-\frac{1}{2},0) + \frac{1}{2}\max(x-\frac{3}{4},0)
\end{aligned}
$$

Bonus question: Can you build the same network using three layers and 6 ReLUs?

$$
\begin{aligned}
z &= \max(x,0) + \max(-x,0) \\
y &= \frac{1}{4}\max(z,0) + \frac{1}{2}\max(z-\frac{1}{4},0) \\
&+ \frac{1}{2}\max(z-\frac{1}{2},0) + \frac{1}{2}\max(z-\frac{3}{4},0)
\end{aligned}
$$

4 Cosine I

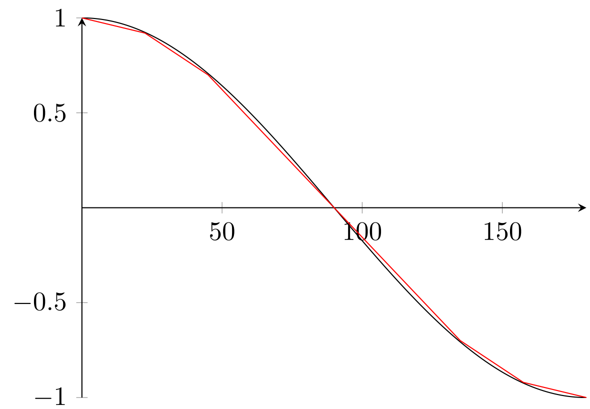
Build a two layer ReLU-network that approximates the cosine function $y = \cos(x)$ in the range $[0°, 90°]$ using 3 hidden units. *Use* $\cos(22.5°) \approx 0.92$*,* $\cos(45°) \approx 0.7$*,* $\cos(90°) = 0$ *and a calculator.*



$$
\begin{aligned}
cos90(x) =& 1 - ReLU(x)\frac{0.08}{22.5} \\
& - ReLU(x - 22.5)\frac{0.14}{22.5} \\
& - ReLU(x - 45)\frac{0.13}{22.5}
\end{aligned}
$$

5 Cosine II

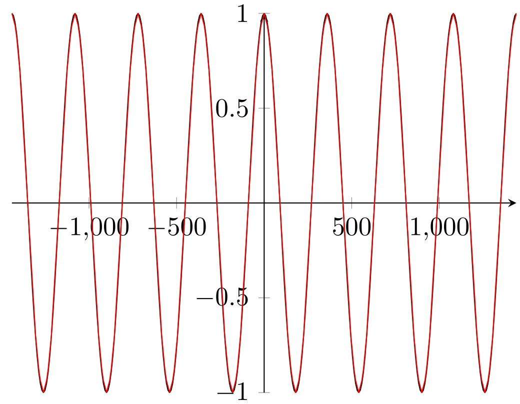
Build a two layer ReLU-network that approximates the cosine function $y = \cos(x)$ in the range $[0°, 180°]$ using 5 hidden units. *You may use the above function, no need to copy the network.*



$$
\begin{aligned}
cos180(x) =& cos90(x) \\
& + ReLU(x-135)\frac{0.13}{22.5} \\
& + ReLU(x-157.5)\frac{0.14}{22.5}
\end{aligned}
$$

6 Cosine III

Build a multi layer ReLU-network that approximates the cosine function $y = \cos(x)$ in the range $[-1440°, 1440°]$ using at most 5 hidden units per layer. *Hint: You can reuse previous networks directly, no need to copy them here. E.g.* $abs(x)$.



$$z_1 = abs(x)$$
$$z_2 = abs(720 - z_1)$$
$$z_3 = abs(360 - z_2)$$
$$z_4 = 180 - abs(180 - z_3)$$
$$y = cos180(z_4)$$

Now we split the entire range of the cosine into 16 parts and feed them into $cos180$. The point $x = 0$ maps to $z_4 = 0$ and $x = 90$ to $z_4 = 90$.

Bonus question: How many hidden units would you need to compute this using a two layer network? 80 units. The above 6 layer network only uses 15 units.